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Quantum Properties of Black Holes: Further Understanding the Double Cone Spacetime

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This paper presents a further examination and confirmation of geodesic completeness of the double cone spacetime, a black hole topology created by Douglass Stanford and Stephen Shenker at Stanford University. This topology is an essential part of a calculation which will determine the quantum properties of black holes and could ultimately help show how quantum mechanics and general relativity fit together.

I. INTRODUCTION

The tension which arises at the intersection of general relativity and quantum mechanics seems to present a fundamental issue in these theories. One case in which this tension is demonstrated is the black hole information problem [1].

This problem can be seen in the following situation. If we were to throw any object—say, a cat—into a black hole, the information of this cat would be stored in the black hole. Steven Hawking, using principles of general relativity, argued that black holes are all slowly evaporating into what we call Hawking radiation, which is essentially just heat [2]. This heat does not contain the information which was stored in the black hole (for example, the information of the cat), so it seems that the information is disappearing. However, according to the principles of quantum mechanics, such information cannot be destroyed.

In an attempt to better understand how these two theories may fit together, Douglass Stanford and Stephen Shenker of Stanford University set out to prove that black holes do, in fact, behave as quantum systems. One way to support this idea is to show that black holes have discrete energy states. Stanford and Shenker created a topology known as the double cone spacetime, which is a key part of the calculation of these energy states [3].

It was initially unclear whether the topology was sufficiently complete to give an accurate result to the calculation of energy states. This paper presents a further examination of this double cone spacetime through a mathematical shift into complex space.

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II. THEORY

A. Stanford and Shenker's Calculation

The calculation of each energy level would be difficult. It is easier to instead calculate the average of two sums of the energy levels by the formula

$$\sum_{E,E'} e^{iT(E-E')}. \tag{1}$$

(1) can be computed by a path integral that sums over spacetimes having two boundaries and where time has period T in each boundary:

$$\int_G e^{iS/\hbar}\mathcal{D}g. \tag{2}$$

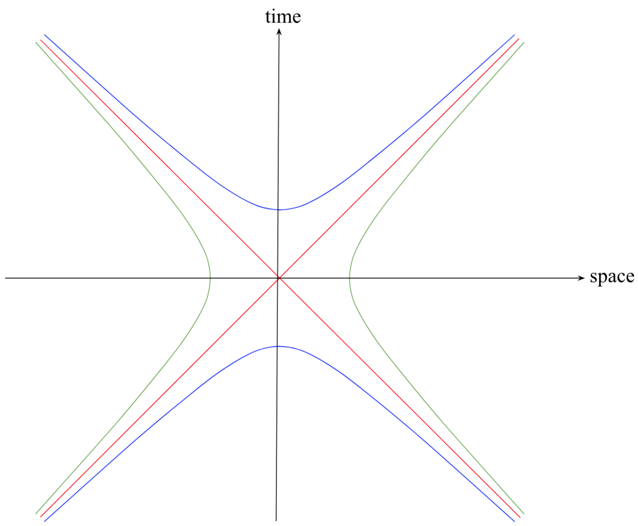
In (2), S is the action for general relativity, $\mathcal{D}g$ is the standard measure on the space of geometries, and G represents all metrics that satisfy the above boundary conditions. Using the stationary phase approximation, this path integral can be approximated by

$$e^{iS(g_0)/\hbar}. \tag{3}$$

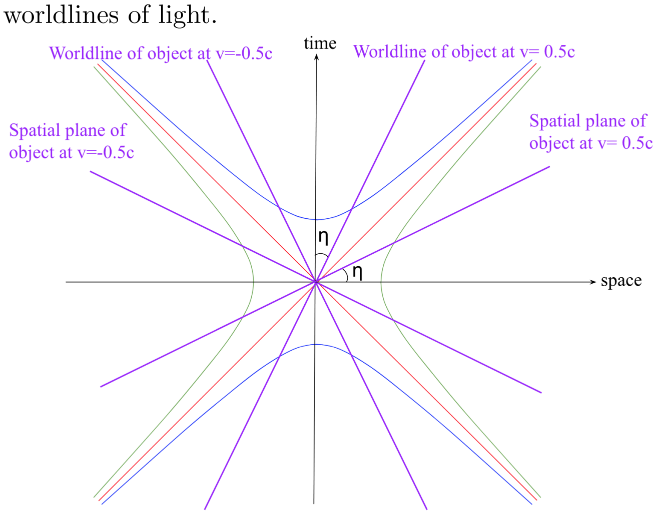
In (3), $S(g_0)$ is a function of g_0, the metric at which S is stationary. This metric can be found using classical solutions of Einstein's equations that satisfy the above boundary conditions.

B. The double cone spacetime

The topology which gives the metric g_0 is called the double cone spacetime. In order to understand this topology, we must first understand the Minkowski spacetime diagram.

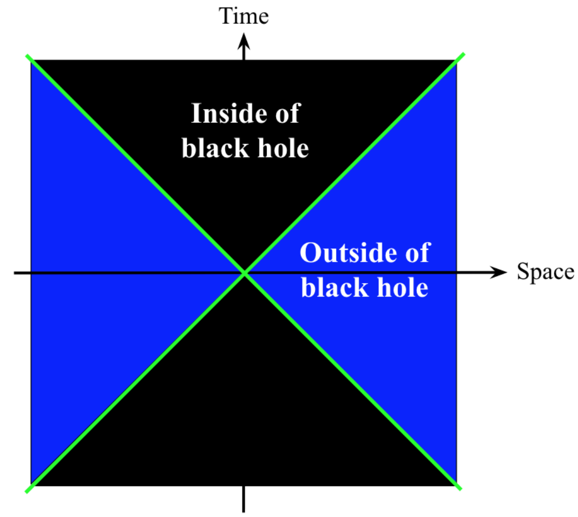


On this diagram, an object at a constant proper distance from the origin (i.e. an object which is not moving in the object's own frame of reference) will follow a hyperbolic path (for example, the green line) rather than a straight path. This is because the frame of reference in a Minkowski diagram is a freely falling frame—one that is actually accelerating in the aforementioned object's reference frame. A line which remains at a constant proper time away from the origin will follow a similar hyperbolic path, but it will be in the top or bottom quadrant rather than the left or right. The worldline of an object in free fall with zero velocity relative to the reference frame of the diagram will be a vertical line. As this relative velocity increases, the worldline will rotate along the hyperbola. The angle of rotation is called the boost parameter, η. It is important to note that as the boost parameter approaches infinity, the worldline will approach (but never cross) the red lines, which are the worldlines of light.

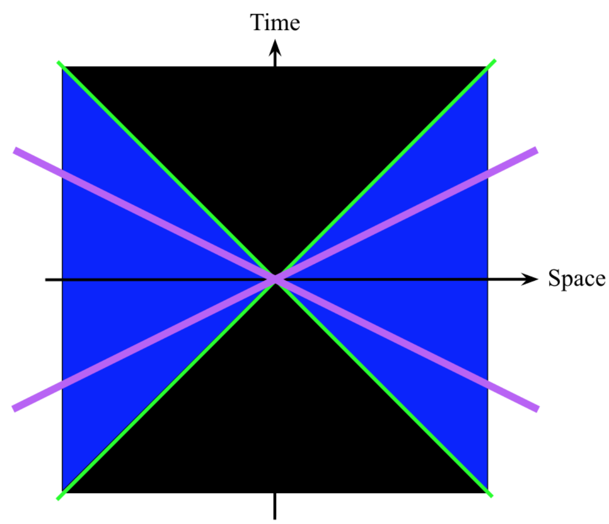


Another important concept is the spatial planes. As the velocity of the object increases, the spatial plane of the object will be offset from the x axis by this same boost parameter.

A black hole on a similar spacetime diagram would look like this [4].

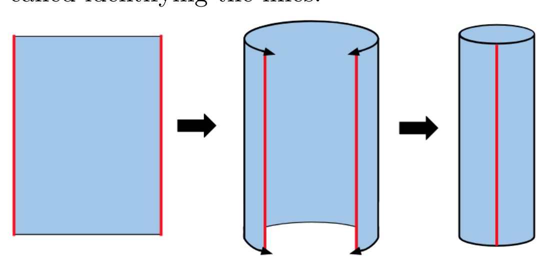


The spatial planes described in the previous paragraph could also be placed on this black hole diagram.

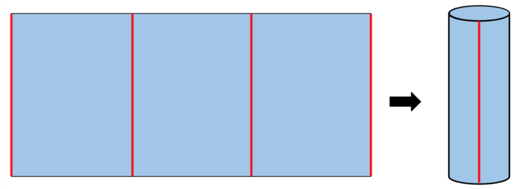


To form the double cone spacetime, we will quotient the blue region of this diagram. The following example provides an intuitive explanation of the quotienting process. (For more information on quotienting, see [5].)

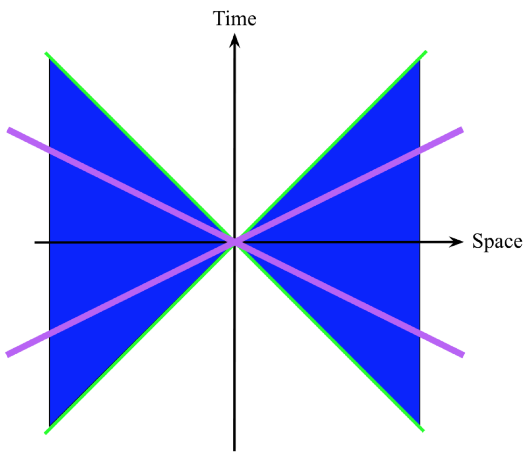
Imagine gluing two ends of a rectangle together to form a cylinder, as shown in the following figure. This process is called identifying the lines.



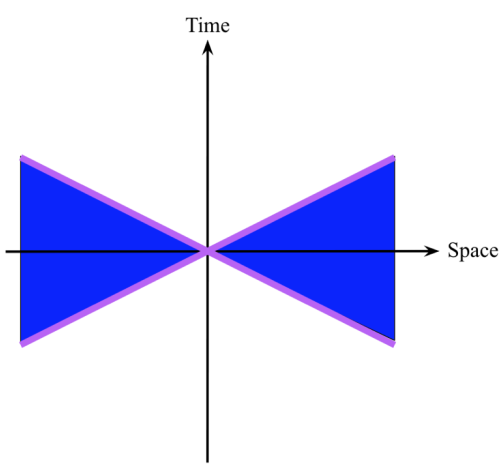
The figure below shows a similar process; we can imagine taking a series of rectangles and folding them into a cylinder of the same circumference as our single-rectangle cylinder. Each subsequent rectangle will create a cylinder which overlaps on top of the original cylinder. We say that the points on these rectangles are "mapped" to points which they lay on top of in the original rectangle (and, subsequently, the original cylinder). This final cylinder is called the quotient.



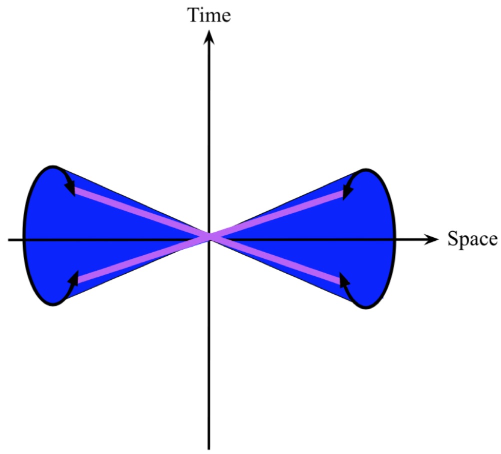
To take the quotient of our black hole geometry, we will first take out the top and bottom quadrants. This is because every point in the region which we quotient must be able to be mapped to the region between the planes of space by the boost parameter.

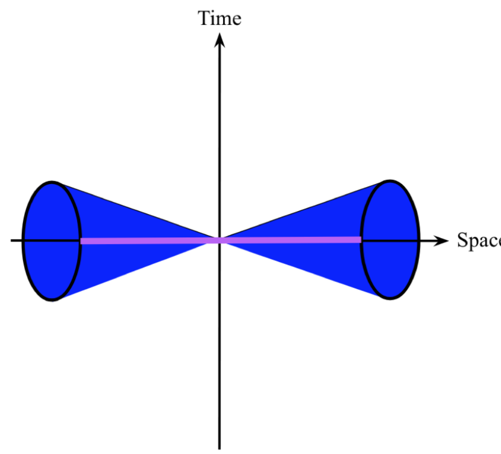


Next, we will take all of the points outside of the two planes of space and map them to points within the region.



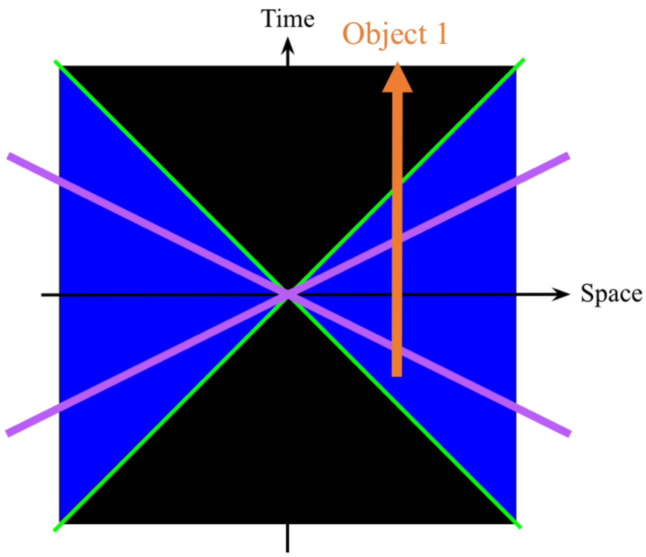
Finally, we will roll this region into the cone by identifying the two planes of space. This creates the double cone spacetime.





C. The question

Each point within the blue region can be mapped to a point on the double cone spacetime. However, an object in free fall that starts outside of the black hole in the flat space (the space before the quotient) will eventually reach the black hole region, which cannot be mapped to any points on the cone. Object 1 on the following diagram follows one possible worldline of a freely falling object.



It is unclear what happens to this object in the double cone spacetime. Therefore, we will be examining what happens to a freely falling object in the double cone spacetime in order to determine if this topology is complete. If the object does not leave the region, we call the space geodesically complete. (See Appendix A for a further explanation of geodesic completeness.) In this case, we can conclude that the double cone spacetime provides a complete picture of the spacetime geometry necessary for the calculation. However, if the object leaves the region after a finite amount of proper time, we may need to make modifications—such as the addition of another double cone representing the top and bottom quadrants—to the spacetime in order to get an accurate calculation.

III. METHOD

In order to find the solution, we must shift the graph slightly into the four dimensional complex plane before taking the quotient. If we fix the boundary conditions to be the same as the boundary conditions in real space, this shifted geometry will give us the same answer to the calculation of S_0 (according to the Cauchy integral theorem).

IV. CALCULATIONS

Coordinates of space (x) and time (t) can be translated into coordinates of the boost parameter (η) and rho (ρ) by the following equations:

$$x = \rho cosh(\eta) \tag{4}$$

$$t = \rho sinh(\eta). \tag{5}$$

To shift the graph slightly into the i dimension, we must add $i\epsilon$ to ρ. To maintain proper boundary conditions, we must also make ϵ dependent on ρ such that $\lim_{\rho\to\infty} \epsilon = 0$.

We are primarily interested in the limits of the real and imaginary parts of x and t at the boundaries of the right region, as this will tell us what happens to the worldline of the previously mentioned object as it continues in free fall. (See Appendix B for calculations of the left quadrant.) At the upper limit of the right quadrant, η approaches ∞ and $\rho + i\epsilon$ approaches $i\epsilon$. Knowing this, we can calculate the upper boundary limits of x and t:

$$x = \lim_{\rho+i\epsilon\to i\epsilon,\ \eta\to\infty} (\rho + i\epsilon)cosh(\eta) = i\infty \tag{6}$$

$$t = \lim_{\rho+i\epsilon\to i\epsilon,\ \eta\to\infty} (\rho + i\epsilon)sinh(\eta) = i\infty. \tag{7}$$

At the lower boundary of the right quadrant, η approaches $-\infty$ and $\rho + i\epsilon$ approaches $i\epsilon$. Knowing this, we can calculate the lower boundary limits of x and t:

$$x = \lim_{\rho+i\epsilon\to i\epsilon,\ \eta\to-\infty} (\rho + i\epsilon)cosh(\eta) = i\infty \tag{8}$$

$$t = \lim_{\rho+i\epsilon\to i\epsilon,\ \eta\to-\infty} (\rho + i\epsilon)sinh(\eta) = -i\infty. \tag{9}$$

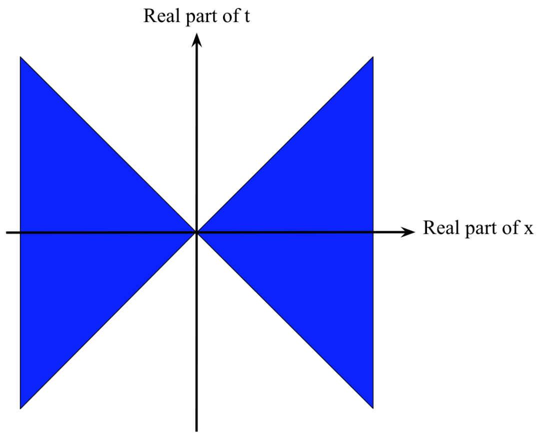
Because x goes to $i\infty$ at both the upper and lower boundaries, we need to calculate the limit of x at a different point to find the range of x. It turns out that the minimum value of x occurs at the intersection of the upper and lower boundaries, where η approaches 0 and $\rho + i\epsilon$ approaches $i\epsilon$

$$x = \lim_{\rho+i\epsilon\to i\epsilon,\ \eta\to 0} (\rho + i\epsilon)cosh(\eta) = i\epsilon. \tag{10}$$

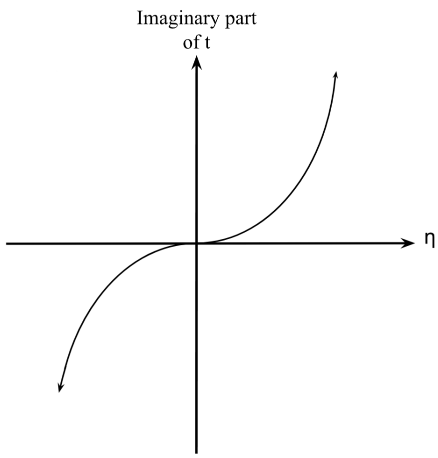
To summarize, the limits of x are $[i\epsilon, i\infty]$, and the limits of t are $[-i\infty, i\infty]$. This is equivalent to saying the limits of the imaginary part of x are $[-\infty, -\epsilon]$, the limits of the imaginary part of t are $[-\infty, \infty]$, and the real parts of x and t are all finite—they have a value of 0 at the upper and lower boundaries of the right region.

V. RESULTS

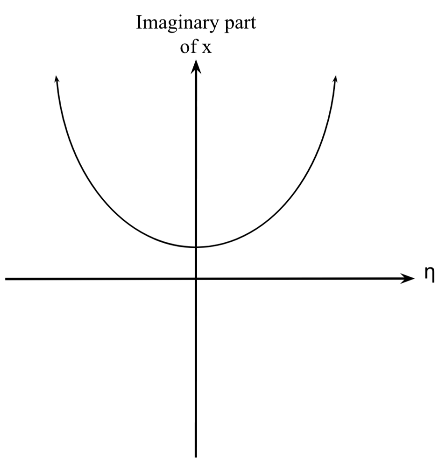
The above calculations tell us that the worldline of an object in free fall will no longer leave the right region. This is because the imaginary parts of space and time are no longer finite after the shift. While it is difficult to imagine this four dimensional space, we can gain a more intuitive understanding of the geometry by examining the parts.



The real parts of space and time remain finite after the shift.



The imaginary part of t approaches ∞ as η approaches ∞, which is at the event horizon of the black hole region.



The imaginary part of x also approaches ∞ as η approaches ∞.

We can conclude that the double cone spacetime gives a complete picture of what happens to an object in free fall—we say it is geodesically complete.

VI. CONCLUSIONS

The double cone spacetime created by Douglass Stanford and Stephen Shenker is a geodesically complete topology and is thus valid for use in the calculation of quantum properties of black holes.

ACKNOWLEDGMENTS

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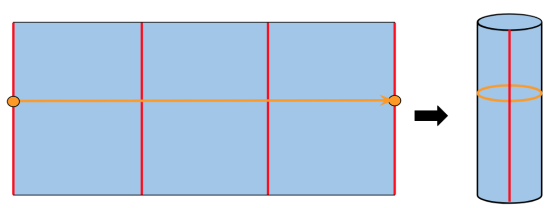
Appendix A: Geodesic Completeness

The concept of geodesic completeness can be demonstrated through the previous example of quotiented rectangles. A geodesic is the shortest possible line between two points. On this series of rectangles, we will assign point 1 to be at the beginning of the first rectangle and point 2 to be at the end of the last rectangle. A geodesic between these two points would look like the orange line on the following diagram.



After we take the quotient of the rectangles, the same line will still exist, it will just be mapped to points on the cylinder. We can then say that this cylinder is geodesically complete because the full geodesic still exists on the cylinder.

It is important to note that, in our black hole geometry, we do not have a finite plane like the series of rectangles. Therefore, a geodesic across the space will be infinite. To understand the concept of geodesic completeness where the geodesic is infinite, we can imagine an infinite number of rectangles in the above example. When these rectangles are quotiented, there are an infinite number of rectangles wrapped around the cylinder, and the infinite geodesic is still present.

Appendix B: Left Region Calculations

At the upper limit of the left quadrant, η approaches $-\infty$ and $\rho + i\epsilon$ approaches $i\epsilon$. Knowing this, we can calculate the upper boundary limits of x and t:

$$x = \lim_{\rho+i\epsilon \to i\epsilon,\, \eta \to -\infty} (\rho + i\epsilon) cosh(\eta) = i\infty \tag{B1}$$

$$t = \lim_{\rho+i\epsilon \to i\epsilon,\, \eta \to -\infty} (\rho + i\epsilon) sinh(\eta) = -i\infty. \tag{B2}$$

At the lower boundary of the left quadrant, η approaches ∞ and $\rho + i\epsilon$ approaches $i\epsilon$. Knowing this, we can calculate the lower boundary limits of x and t:

$$x = \lim_{\rho+i\epsilon \to i\epsilon,\, \eta \to \infty} (\rho + i\epsilon) cosh(\eta) = i\infty \tag{B3}$$

$$t = \lim_{\rho+i\epsilon \to i\epsilon,\, \eta \to \infty} (\rho + i\epsilon) sinh(\eta) = i\infty. \tag{B4}$$

As in the right region, we need to calculate the limit of x at the intersection of the upper and lower boundaries, where η approaches 0 and $\rho + i\epsilon$ approaches $i\epsilon$:

$$x = \lim_{\rho+i\epsilon \to i\epsilon,\, \eta \to 0} (\rho + i\epsilon) cosh(\eta) = i\epsilon. \tag{B5}$$

The left region turns out to have the same limits of t as the right region. The limits of x change to $[-i\infty, i\epsilon]$, but this would still give us a geodesically complete region—the imaginary part of x would just bend in the opposite direction.

[1] J. Polchinski, The Black Hole Information Problem, in *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings* (2017) pp. 353–397, arXiv:1609.04036 [hep-th].

[2] S. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. **43**, 199 (1975), [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[3] P. Saad, S. H. Shenker, and D. Stanford, A semiclassical ramp in SYK and in gravity, (2018), arXiv:1806.06840 [hep-th].

[4] D. Marolf, Notes on Relativity and Cosmology (2003), http://web.physics.ucsb.edu/∼marolf/MasterNotes.pdf.

[5] M. Hitchman, Quotienting spaces (2018 (accessed August 10, 2020)), https://mphitchman.com/geometry/section7-7.html.